The compressibility of rotating black holes in D-dimensions

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Abstract

Treating the cosmological constant as a pressure, in the context of black hole thermodynamics, a thermodynamic volume for the black hole can be defined as being the thermodynamic variable conjugate to the pressure, in the sense of a Legendre transform. The thermodynamic volume is explicitly calculated, as the Legendre transform of the pressure in the enthalpy, for a rotating asymptotically anti-de Sitter Myers-Perry black hole in D space-time dimensions. The volume obtained is shown to agree with previous calculations using the Smarr relation. The compressibility is calculated and shown to be non-negative and bounded.

Taking the limit of zero cosmological constant, the compressibility of a rotating black hole in asymptotically flat space-times is determined and the corresponding speed of sound computed. The latter is bounded above and has an elegant expression purely in terms of the angular momenta, in the form of quartic and quadratic Casimirs of the rotation group, $SO(D-1)$.

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1 Introduction

The thermodynamics of black holes has been an active area of research ever since Bekenstein and Hawking's seminal papers on the entropy and temperature associated with the event horizon of a black hole, [1, 2]. Recently the rôle of pressure and volume has come under scrutiny in this context. It was pointed out in [3] that the presence of a cosmological constant, Λ, spoils the otherwise successful Smarr relation [4] and a remedy was proposed: to raise Λ to the status of a thermodynamic variable, on a par with the temperature, while at the same time the black hole mass should be interpreted as the thermodynamic potential associated with the enthalpy, rather than the heretofore more usual interpretation of internal energy. It is then very natural to identify Λ as being proportional to a pressure and the thermodynamic variable conjugate to the pressure can be interpreted as a volume for the black hole [5]. The idea of promoting Λ to the status of a thermodynamic variable is not new, [6]-[11], but it is only recently that a volume has entered the picture in this context. For a rotating black hole in four dimensions this thermodynamic volume does not have any obvious relation to any geometric volume, though they agree if the black hole is not rotating, [12]. Nevertheless, with the volume included, there is a remarkable similarity between the black hole equation of state and that of a Van der Waals gas, [12]-[17].

An important physical quantity in any thermodynamics system that behaves like a gas is the compressibility, which was investigated for rotating, asymptotically anti-de Sitter (AdS) black holes in 4-dimensions in [18], including asymptotically flat space-times as a limiting case. In this paper the investigation of the compressibility of asymptotically AdS black holes is extended to dimensions greater than four. To that end we first derive the compressibility of a rotating asymptotically AdS Myers-Perry black hole in D space-time dimensions. Our aim is to derive the compressibility and the speed of sound for asymptotically flat Myers-Perry black holes, but we must include a non-zero Λ in order to obtain the volume and the compressibility before taking the limit $\Lambda \to 0$. In this limit the expressions simplify considerably and the compressibility and the speed of sound can be expressed rather compactly in terms of the quadratic and quartic Casimirs of $SO(D-1)$ associated with the angular momenta of the black hole.

It turns out, as was emphasised in [12], that it is crucial that the black hole be rotating: if it is not rotating the entropy S and the volume V are both functions of the event horizon radius r_h only — they are not independent and cannot be considered to be independent thermodynamic variables. They become independent only when the black hole rotates, the Legendre transform is not well defined in the limit of zero rotation.

We restrict the analysis here to asymptotically AdS and asymptotically flat space-times. The thermodynamics of black holes in $\Lambda > 0$ space-times is a notoriously delicate issue. First steps in understanding the rôle of a thermodynamic volume of black holes in this case were taken in [20] but unresolved issues remain, these are left for future work and are avoided here by restricting to $\Lambda \leq 0$.

In section §2 we summarise the relevant features of asymptotically AdS Myers-Perry black holes, determine the thermodynamic volume and describe the compressibility, the main result is the compressibility in equation (22). In §3 the $\Lambda \to 0$, asymptotically flat, limit is investigated; the compressibility, given in (33) and speed of sound in (39), are derived and physical implications are discussed, particularly in relation to ultra-spinning black holes in $D > 4$. The conclusions discuss some implications of the results and possible future directions. Finally some technical details are confined to two appendices.

2 AdS Myers-Perry black holes

Rotating black holes in D-dimensions must be treated slightly differently for even and odd D because the rotation group $SO(D-1)$, acting on the event horizon which is assumed to have the topology of a $(D-2)$-dimensional sphere, has different characterisations of angular momenta in the even and odd dimensional cases. The Cartan sub-algebra has dimension $\frac{D-2}{2}$ for even D and $\frac{D-1}{2}$ for odd D so a general state of rotation is specified by $\frac{D-2}{2}$ independent angular momenta in even D and $\frac{D-1}{2}$ in odd D. Let $p = \left\lfloor \frac{D-1}{2} \right\rfloor$, the integral part of $\frac{D-1}{2}$, be the dimension of the Cartan sub-algebra of $SO(D-1)$, then there are p independent angular momenta J_i, $i = 1, \dots, p$. It is notationally convenient to introduce a parameter $\epsilon = 1$ for even D and $\epsilon = 0$ for odd D, so

$$p = \frac{D-1-\epsilon}{2}. \tag{1}$$

In this notation the unit $(D-2)$-dimensional sphere can be described in terms of Cartesian co-ordinates x_a in $\mathbf{R}^{D-1}$ by

$$\sum_{a=1}^{D-1} x_a^2 = 1, \tag{2}$$

and we can write this as

$$\sum_{i=1}^{p} \rho_i^2 + \epsilon y^2 = 1, \tag{3}$$

where $x_{2i-1} + i x_{2i} = \rho_i e^{i\phi_i}$, $i = 1, \dots, p$, are complex co-ordinates for both the even and odd cases while $y = x_{D-1}$ is only necessary for even D.

ρ_i, ϕ_i and y are then (redundant) co-ordinates that can be used to parameterise the sphere and, for the black hole, J_i are angular momenta in the (x_{2i-1}, x_{2i})-plane.

Myers-Perry black holes in D-dimensions with a cosmological constant, Λ, were constructed in [21]: they are solutions of Einstein's equations with Ricci tensor[1]

$$R_{\mu\nu} = \frac{2\Lambda}{(D-2)} g_{\mu\nu}. \tag{4}$$

We shall focus on $\Lambda \leq 0$ here, as the thermodynamics is then better understood and for notational convenience we define

$$\lambda = -\frac{2\Lambda}{(D-1)(D-2)} \geq 0. \tag{5}$$

The line element in [21] can then be expressed, in Boyer-Linquist co-ordinates, as[2]

$$\begin{aligned} ds^2 \;=\; & -W(1+\lambda r^2)dt^2 + \frac{2\mu}{U}\left(W dt - \sum_{i=1}^{p} \frac{a_i \rho_i^2 d\phi_i}{1-\lambda a_i^2} \right)^2 \\ & + \left(\frac{U}{Z-2\mu} \right) dr^2 + \epsilon\, r^2 dy^2 + \sum_{i=1}^{p} \left(\frac{r^2 + a_i^2}{1-\lambda a_i^2} \right) (d\rho_i^2 + \rho_i^2 d\phi_i^2) \\ & - \frac{\lambda}{W(1+\lambda r^2)} \left(\sum_{i=1}^{p} \left(\frac{r^2+a_i^2}{1-\lambda a_i^2} \right) \rho_i d\rho_i + \epsilon r^2 y dy \right)^2 , \end{aligned} \tag{6}$$

[1] We use units with Newton's constant and the speed of light set to unity, $G = c^2 = 1$.

[2] The form given here differs slightly from that in [21] in that our ordinates, t and ϕ_i, are related to those of [21], τ and φ_i, by $d\tau = dt$ and $d\phi_i = d\varphi_i - \lambda a_i dt$.

where the functions W, Z and U are

$$\begin{aligned} W &= \epsilon y^2 + \sum_{i=1}^{p} \frac{\rho_i^2}{1-\lambda a_i^2} \\ Z &= \frac{(1+\lambda r^2)}{r^{2-\epsilon}} \prod_{i=1}^{p} (r^2 + a_i^2) \\ U &= \frac{Z}{1+\lambda r^2} \left(1 - \sum_{i=1}^{p} \frac{a_i^2 \rho_i^2}{r^2 + a_i^2} \right). \end{aligned} \tag{7}$$

The a_i are rotation parameters in the (x_{2i-1}, x_{2i})-plane, restricted to $a_i^2 < 1/\lambda$, and μ is a mass parameter.

Many of the properties of the space-time with line element (7) were described in [21]. There is an event horizon at r_h, the largest root of $Z - 2\mu = 0$, so

$$\mu = \frac{(1+\lambda r_h^2)}{2 r_h^{2-\epsilon}} \prod_{i=1}^{p} (r_h^2 + a_i^2), \tag{8}$$

with area

$$\mathcal{A}_h = \frac{\varpi}{r_h^{1-\epsilon}} \prod_{i=1}^{p} \frac{r_h^2 + a_i^2}{1 - \lambda a_i^2}\,, \tag{9}$$

where ϖ is is the volume of the round unit $(D-2)$-sphere,

$$\varpi = \frac{2\pi^{\frac{(D-1)}{2}}}{\Gamma\left(\frac{D-1}{2}\right)}\,. \tag{10}$$

The Bekenstein-Hawking entropy is

$$S = \frac{\varpi}{4 r_h^{1-\epsilon}} \prod_{i=1}^{p} \frac{r_h^2 + a_i^2}{1 - \lambda a_i^2} \tag{11}$$

and the Hawking temperature is, with $\hbar = 1$,

$$T = \frac{r_h}{2\pi}(1+\lambda r_h^2) \sum_{i=1}^{p} \frac{1}{r_h^2 + a_i^2} + \frac{(2-\epsilon)(\epsilon \lambda r_h^2 - 1)}{4\pi r_h}. \tag{12}$$

The angular momenta and the ADM mass, M, of the black hole are related to the metric parameters via

$$J_i = \frac{\mu\, \varpi a_i}{4\pi(1-\lambda a_i^2) \prod_j (1-\lambda a_j^2)}, \tag{13}$$

$$
\begin{aligned}
M &= \frac{\mu\,\varpi}{8\pi\prod_j(1-\lambda a_j^2)}\left(D-2+2\lambda\sum_{i=1}^{p}\frac{a_i^2}{1-\lambda a_i^2}\right) \\
&= \frac{(D-2)\mu\,\varpi}{8\pi\prod_j(1-\lambda a_j^2)}+\lambda\sum_{i=1}^{p}J_i a_i,
\end{aligned}
\tag{14}
$$

while the angular velocities are

$$
\Omega_i = \frac{(1+\lambda r_h^2)a_i}{(r_h^2+a_i^2)}. \tag{15}
$$

It was argued in [3] that, in the presence of a cosmological constant, the correct thermodynamic interpretation of the black hole mass is that it is the enthalpy of the system

$$
M = H(S, P, J), \tag{16}
$$

where J stands for all the J_i collectively, and the pressure is

$$
P = -\frac{\Lambda}{8\pi} = \frac{(D-1)(D-2)\lambda}{16\pi}. \tag{17}
$$

The thermodynamic volume, V, is defined as the variable thermodynamically conjugate to P [5, 12],

$$
V = \left.\frac{\partial M}{\partial P}\right|_{S,J} = \frac{16\pi}{(D-1)(D-2)}\left.\frac{\partial M}{\partial \lambda}\right|_{S,J}. \tag{18}
$$

Details of the calculation of the thermodynamic volume by this technique are given in appendix A and here we quote the result (44)

$$
\begin{aligned}
V &= \frac{r_h\mathcal{A}_h}{D-1}\left\{1+\frac{(1+\lambda r_h^2)}{(D-2)r_h^2}\sum_{i=1}^{p}\frac{a_i^2}{(1-\lambda a_i^2)}\right\} \\
&= \frac{r_h\mathcal{A}_h}{D-1}+\frac{8\pi}{(D-2)(D-1)}\sum_{i=1}^{p}a_i J_i.
\end{aligned}
\tag{19}
$$

With the substitution $\lambda \to g^2$ this agrees with the result [19] for the black hole volume, derived from the assumption that the Smarr relation,

$$
(D-3)M = (D-2)TS + (D-2)\sum_{i=1}^{p}\Omega_i J_i - 2PV, \tag{20}
$$

holds. For $D=4$ it reproduces the corresponding expression in [12]. With the substitution $\lambda \to -g^2$, (20) agrees with the black hole thermodynamic volume quoted in [20] for $\Lambda > 0$, again determined by assuming the Smarr relation holds, but avoiding the complication of the existence of the cosmological horizon that is present in this case.

It is now possible to define the adiabatic compressibility of the black hole [18] as

$$\kappa = -\frac{1}{V}\left.\frac{\partial V}{\partial P}\right|_{S,J}. \tag{21}$$

With the explicit form of the thermodynamic volume in (20) the compressibility can be computed using the technique outlined in appendix B: it evaluates to

$$\kappa = \frac{16\pi(1+\lambda r_h^2)}{(D-1)(D-2)^2}\frac{\left\{\sum_{i=1}^p \frac{a_i^4}{1-\lambda^2 a_i^4} - \frac{1}{(D-2)}\left(\sum_{i=1}^p \frac{a_i^2}{1-\lambda a_i^2}\right)\left(\sum_{i=1}^p \frac{a_i^2}{1+\lambda a_i^2}\right)\right\}}{\left\{r_h^2 + \frac{(1+\lambda r_h^2)}{(D-2)}\sum_{i=1}^p \frac{a_i^2}{1-\lambda a_i^2}\right\}\left\{1 - \frac{2\lambda}{(D-2)}\sum_{i=1}^p \frac{a_i^2}{1+\lambda a_i^2}\right\}}. \tag{22}$$

It is shown in the appendix that $\kappa \geq 0$. It is also not difficult to prove that it is bounded above for $\lambda > 0$: to see this first observe that the denominator never vanishes because

$$D-2-2\lambda\sum_{i=1}^p \frac{a_i^2}{1+\lambda a_i^2} = \epsilon - 1 + 2\sum_{i=1}^p \frac{1}{1+\lambda a_i^2} \geq \frac{D-3-\epsilon}{2}, \tag{23}$$

with equality when all the a_i achieve the maximum value, $a_1^2 = \cdots a_p^2 = \frac{1}{\lambda}$. The numerator can diverge though, if any or all of the a_i^2 approach $\frac{1}{\lambda}$, but when this happens the first factor in curly brackets in the denominator also diverges, the singularities cancel and κ remains finite. For example, if m of the a_i^2 approach $1/\lambda$, with $1 \leq m \leq p$, and the others are all zero, then

$$\kappa \to \frac{8\pi}{(D-1)(D-2)\lambda} = \frac{1}{2P}, \tag{24}$$

reflecting the fact that $V \propto \frac{1}{\sqrt{P}}$ in this limit.

A thermodynamic speed of sound, c_s, can be defined by using the homogeneous density,

$$\rho = \frac{M}{V} = \frac{(D-1)(D-2)(1+\lambda r_h^2)}{16\pi r_h^2}\frac{\left(1+\frac{2\lambda}{D-2}\sum_{i=1}^p \frac{a_i^2}{1-\lambda a_i^2}\right)}{\left(1+\frac{(1+\lambda r_h^2)}{(D-2)\, r_h^2}\sum_{i=1}^p \frac{a_i^2}{1-\lambda a_i^2}\right)}. \tag{25}$$

The usual thermodynamic relation can then be used to obtain a speed of sound,

$$\frac{1}{c_s^2} = \left.\frac{\partial\rho}{\partial P}\right|_{S,J} = 1 + \rho\kappa \geq 1. \tag{26}$$

so $0 \leq c_s^2 \leq 1$. Again, for example, taking m of the a_i^2 to approach $1/\lambda$ with the others all zero,

$$\rho \to \frac{(D-1)(D-2)\lambda}{8\pi}, \tag{27}$$

and $\rho\kappa \to 1$, giving $c_s^2 = \frac{1}{2}$.

3 Isentropic processes in asymptotically flat Myers-Perry space-times

When the cosmological constant vanishes many of the expressions in the previous section simplify considerably. In particular

$$\begin{aligned} M &= \frac{(D-2)\varpi\mu}{8\pi}, \qquad S = \frac{\varpi}{4r^{1-\epsilon}}\prod_{i=1}^{p}(r_h^2+a_i^2) = \frac{4\pi}{D-2}Mr_h \\ J_i &= \frac{2Ma_i}{D-2}, \qquad \Omega_i = \frac{a_i}{r_h^2+a_i^2}. \end{aligned} \tag{28}$$

In this section we shall focus on isentropic processes, for which it is convenient to define the dimensionless angular momenta

$$\mathcal{J}_i = \frac{2\pi J_i}{S} = \frac{a_i}{r_h} \tag{29}$$

in terms of which the mass is

$$M = \frac{(D-2)}{16\pi}\varpi r_h^{D-3}\prod_{i=1}^{p}(1+\mathcal{J}_i^2), \tag{30}$$

and the entropy is

$$S = \frac{\varpi}{4}r_h^{D-2}\prod_{i=1}^{p}(1+\mathcal{J}_i^2). \tag{31}$$

The thermodynamic volume is

$$V = \frac{r_h\mathcal{A}_h}{D-1}\left(1+\frac{\sum_i\mathcal{J}_i^2}{D-2}\right) = V_0\prod_{i=1}^{p}(1+\mathcal{J}_i^2)\left(1+\frac{\sum_i\mathcal{J}_i^2}{D-2}\right), \tag{32}$$

where $V_0 = \frac{\varpi r_h^{D-1}}{D-1}$ is the volume of an ordinary $D-1$ dimensional sphere of radius r_h.

The $\lambda \to 0$ limit of (22) is finite:

$$\kappa = \frac{16\pi r_h^2}{(D-1)(D-2)^2} \left\{ \frac{(D-2)\sum_i \mathcal{J}_i^4 - (\sum_i \mathcal{J}_i^2)^2}{D-2+\sum_i \mathcal{J}_i^2} \right\}, \tag{33}$$

with

$$r_h^2 = \left\{ \frac{4S}{\varpi \prod_i (1+\mathcal{J}_i^2)} \right\}^{\frac{2}{D-2}}. \tag{34}$$

Thus at fixed S the compressibility is simply expressible entirely in terms of quadratic and quartic Casimirs of $SO(D-1)$.

For $D \geq 4$, κ is positive, and it is identically zero in $D=3$: this latter result is in intuitive accord with the fact that gravity has no dynamics in the bulk in 3-dimensions, all of the interesting physics is in the boundary conditions.

To understand κ fully, it is necessary to take account of the constraints imposed by the condition that $T \geq 0$. For $\lambda = 0$

$$T = \frac{1}{2\pi r_h}\left(\sum_{i=1}^{p} \frac{1}{1+\mathcal{J}_i^2} - 1 + \frac{\epsilon}{2} \right) \qquad \Rightarrow \qquad \sum_{i=1}^{p} \frac{1}{1+\mathcal{J}_i^2} \geq 1 - \frac{\epsilon}{2}. \tag{35}$$

For $D > 4$ it is possible for some of the $\mathcal{J}_i$ to tend to infinity, but not all of them — the well known phenomenon of ultra-spinning black holes.[3]

Equation (35) says that the locus of allowed temperatures is thus bounded by hyperbolae in $\mathcal{J}$-space. The case for $D=6$ is plotted in figure 1 below.

[3] Note that, although an ultra-spinning black hole has a large J_i, the corresponding angular velocity need not be large: indeed $\Omega_i \to 0$ as $J_i \to \infty$. The inverse of the isentropic momentum of inertia tensor is

$$\mathcal{I}^{-1}_{ij} = \left.\frac{\partial \Omega_i}{\partial J_j}\right|_S = \frac{1}{M r_h^2}\left\{ \frac{(D-2)}{2}\frac{(1-\mathcal{J}_i^2)}{(1+\mathcal{J}_i^2)^2}\delta_{ij} + \frac{\mathcal{J}_i\mathcal{J}_j}{(1+\mathcal{J}_i^2)(1+\mathcal{J}_j^2)} \right\}. \tag{36}$$

For large $\mathcal{J}_i$, $\mathcal{I}^{-1}$ develops a negative eigenvalue and a negative moment of inertia implies that Ω_i *decreases* as J_i increases. Indeed if one of the $\mathcal{J}_i$ tends to infinity as $\mathcal{J}_i = L \to \infty$, at constant finite S, then $r_h \approx L^{-\frac{2}{D-2}}$, from (31), and the corresponding element of $\mathcal{I}^{-1} \approx -L^{\frac{4}{D-2}-2} \to 0$ as $L \to \infty$, provided $D > 4$. Ultra-spinning black holes do not have large angular momenta because they have large angular velocity, they have large angular momenta because their moment of inertia diverges as $J_i \to \infty$.

This is very similar to plots in [22], except there the J_i are normalised using the appropriate power of the mass, relevant for isenthalpic processes, while here the entropy is used, for isentropic processes.

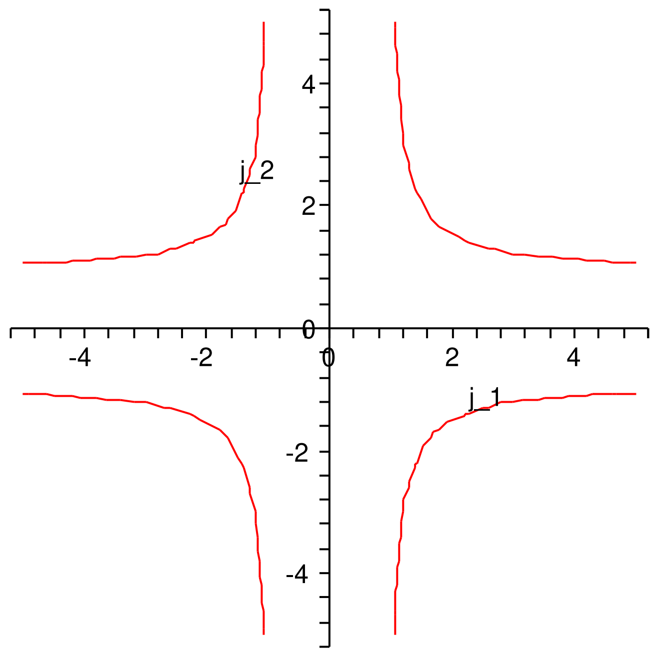

Figure 1: The locus of extremal black holes, $T=0$, for $D=6$. $T>0$ requires the angular momenta to lie inside the region bounded by the hyperbolae.

When all the $\mathcal{J}_i$ are small the compressibility is small and the equation of state is very stiff, the black hole is completely incompressible for $\mathcal{J}_i = 0$. However the compressibility can diverge if some $\mathcal{J}_i$ are kept small while others are sent to infinity. For example, if $\mathcal{J}_1 = \cdots = \mathcal{J}_{p-m} = 0$ and $\mathcal{J}_{p-m+1} = \cdots \mathcal{J}_p = L$, then $T \geq 0$ for $L \to \infty$ provided $m \leq \frac{D-3}{2}$. Also (34) implies that $r_h^2 \propto L^{-\frac{4m}{D-2}}$ so

$$\kappa \sim L^{\frac{2(D-2m-2)}{D-2}}, \tag{37}$$

which diverges if $m < \frac{D-2}{2}$, so κ diverges for $1 \leq m \leq \frac{D-3}{2}$ with this configuration of angular momenta. The divergence is fastest for $m = 1$.

When the compressibility becomes large the black hole equation of state is very soft. For example the compressibility for $D = 6$ is plotted in figure 2 and it grows indefinitely for large angular momenta along either the $\mathcal{J}_1$ or the $\mathcal{J}_2$ axis, *i.e.* $m = 1$.

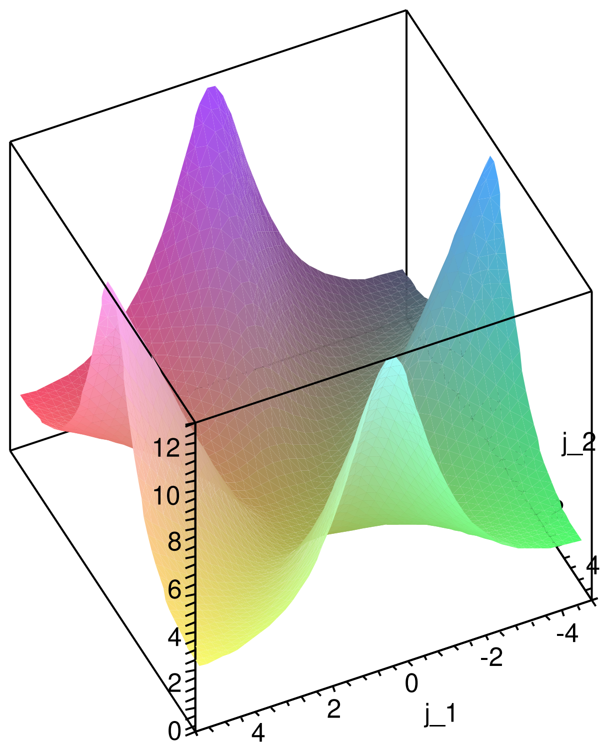

Figure 2: The compressibility of a black hole in D=6 as a function of $\mathcal{J}_1$ and $\mathcal{J}_2$.

It was suggested in [23] that ultra-spinning black holes should be dynamically unstable for large angular momentum, and subsequent numerical and analytical work supports this proposal [24]-[30]. Large compressibility can be taken as a sign of an instability setting in, although there is no indication in equation (33) of a boundary in $\mathcal{J}$-space were a dynamical instability might manifest itself, the expression for the compressibility implies that the instability sets in more quickly when only one angular momentum is taken to be large compatible with the pancake structure of [24].

Again a thermodynamic speed of sound can be defined using

$$\rho = \frac{(D-1)(D-2)^2}{16\pi r_h^2} \frac{1}{(D-2+\sum_{i=1}^{p} \mathcal{J}_i^2)} \tag{38}$$

which gives, with (33) in (26),

$$c_s^2 = \frac{1}{(D-2)} \frac{\left(D-2+\sum_i \mathcal{J}_i^2\right)^2}{\left(D-2+2\sum_i \mathcal{J}_i^2+\sum_i \mathcal{J}_i^4\right)}. \tag{39}$$

It is not immediately clear how the thermodynamic speed of sound might be related to a fluid dynamical speed of sound, but it is noteworthy that the thermodynamic speed of sound is least when the compressibility is greatest, as one would expect for a soft equation of state. Indeed $\frac{1}{D-2} \leq c_s^2 \leq 1$ with $c_s = 1$ for $\mathcal{J}_i = 0$ and $c_s^2 \to \frac{1}{D-2}$ as any one $\mathcal{J}_i \to \infty$ with all others remaining finite. It is possible that the thermodynamic speed of sound is related to the velocity of the kind of waves and vibrations envisaged in figure 6 of [24] associated with the instability of an ultra-spinning black hole, at least for $D > 4$.

4 Conclusions

A cosmological constant spoils the Smarr relation for black hole thermodynamics unless it is given the status of a thermodynamic variable, most naturally interpreted as proportional to a pressure. A consistent interpretation of the ADM mass of the black hole, in terms of thermodynamic potentials, is that it is the enthalpy of the black hole. A thermodynamic volume can then be defined as being the thermodynamic variable conjugate to the pressure, in terms of Legendre transforms.

The main results are the thermodynamic volume (20), computed explicitly as the Legendre transform variable conjugate of the pressure rather than by assuming the Smarr relation, and the compressibility (22) for Myers-Perry black holes in asymptotically anti-de Sitter, D-dimensional, space-time. The corresponding expressions for asymptotically flat space-times then follow easily from the $\Lambda \to 0$ limit, and the corresponding quantities for asymptotically flat Myers-Perry black holes are given in equations (20) and (33) respectively. In addition the speed of sound can be expressed in terms of Casimirs of the rotation group, $SO(D-1)$, and is given in (39).

We emphasise again that it is crucial that the black hole is rotating. It is clear from equations (9), (11) and (20) that, when all $a_i \to 0$, the entropy $S(r_h, \lambda, a_i)$ and the volume $V(r_h, \lambda, a_i)$ are both functions of the event horizon radius r_h only, we then have $V(r_h)$ and $S(r_h)$ and V can be written uniquely as a function of the single variable S: they cannot be considered to be independent thermodynamic variables in this limit. The volume is an independent thermodynamic variable only when the black hole rotates, otherwise the Legendre transform is not well defined, as was first pointed out in [12]. This is reflected in the fact that the isentropic compressibility (22) vanishes as $\mathcal{J}_i \to 0$: fixing S fixes V when the black hole is non-rotating, hence it is incompressible.

The discussion here has been restricted to electrically neutral rotating black holes, leaving open the question of how electric charge might affect compressibility.

It would be very interesting to develop these ideas in the context of positive Λ, but we immediately hit the problem of having two horizons to contend with, a black hole horizon and a cosmological horizon, leading to two, in general different, temperatures and raising the question of how to define thermodynamic potentials for such a system. A preliminary discussion of thermodynamic volumes in this context was given in [20], but only by treating the two horizons as essentially independent and defining two independent volumes. The volume associated with the black hole horizon in [20] was the same as (20), but with $\lambda = -g^2$ negative, and it is perhaps significant in this context that κ in (22) remains positive under this continuation to negative λ, provided $r_h^2 < -\frac{1}{\lambda}$. However a completely consistent integrated thermodynamic treatment of asymptotically de Sitter space-times still eludes us.

A Thermodynamic volume

The thermodynamic volume is calculated by differentiating the mass (14) with respect to λ, keeping the entropy and the angular momenta fixed. To this end we note that (8), (11), (13) and (14) allow us to write

$$J_i = \frac{S}{2\pi r_h} \frac{(1 + \lambda r_h^2)}{(1 - \lambda a_i^2)} a_i \tag{40}$$

and demanding $dJ_i|_S = 0$ then gives

$$da_i = \frac{a_i}{(1+\lambda a_i^2)(1+\lambda r_h^2)} \left\{ (1-\lambda r_h^2)\,(1-\lambda a_i^2)\,\frac{dr_h}{r_h} - (r_h^2 + a_i^2)\,d\lambda \right\}. \quad (41)$$

A second relation between da_i, dr_h and $d\lambda$ follows from $dS|_{J_i} = 0$ in (11), allowing the elimination of da_i to give

$$dr_h = \left(\frac{\sum_i \frac{a_i^2}{1+\lambda a_i^2}}{D - 2 + 2\lambda \sum_i \frac{a_i^2}{1+\lambda a_i^2}} \right) r_h\, d\lambda, \quad (42)$$

and we have all the ingredients necessary to calculate $\left.\frac{\partial}{\partial\lambda}\right|_{S,J}$ acting on any function of λ, r_h and a_i.

The thermodynamic volume is perhaps most easily calculated by combining (8), (11), and the mass in (14), to write

$$M = \frac{S}{4\pi} \frac{(1+\lambda r_h^2)}{r_h} \left(D - 2 + 2\lambda \sum_{i=1}^{p} \frac{a_i^2}{1-\lambda a_i^2} \right). \quad (43)$$

Using this equation (41) and (42) yields the following formula for the volume

$$\begin{aligned} V = \frac{16\pi}{(D-1)(D-2)} \left.\frac{\partial M}{\partial \lambda}\right|_{S,J} &= \frac{4 r_h S}{D-1} \left\{ 1 + \frac{(1+\lambda r_h^2)}{(D-2) r_h^2} \sum_{i=1}^{p} \frac{a_i^2}{(1-\lambda a_i^2)} \right\} \\ &= \frac{4 r_h S}{D-1} + \frac{8\pi}{(D-2)(D-1)} \sum_{i=1}^{p} a_i J_i. \quad (44) \end{aligned}$$

B Compressibility

The compressibility can be evaluated by pushing the analysis of appendix A one step further and calculating

$$\kappa = -\frac{16\pi}{(D-1)(D-2)} \frac{1}{V} \left.\frac{\partial V}{\partial \lambda}\right|_{S,J} \quad (45)$$

A tedious, but straightforward calculation, gives

$$\kappa = \frac{16\pi(1+\lambda r_h^2)}{(D-1)(D-2)^2} \frac{\left\{ \sum_{i=1}^{p} \frac{a_i^4}{1-\lambda^2 a_i^4} - \frac{1}{(D-2)} \left(\sum_{i=1}^{p} \frac{a_i^2}{1-\lambda a_i^2} \right) \left(\sum_{i=1}^{p} \frac{a_i^2}{1+\lambda a_i^2} \right) \right\}}{\left\{ r_h^2 + \frac{(1+\lambda r_h^2)}{(D-2)} \sum_{i=1}^{p} \frac{a_i^2}{1-\lambda a_i^2} \right\} \left\{ 1 - \frac{2\lambda}{(D-2)} \sum_{i=1}^{p} \frac{a_i^2}{1+\lambda a_i^2} \right\}}. \quad (46)$$

We can show that $\kappa \geq 0$. First note that

$$D - 2 - 2\lambda \sum_{i=1}^{p} \frac{a_i^2}{1+\lambda a_i^2} = \epsilon - 1 + 2\sum_{i=1}^{p} \frac{1}{1+\lambda a_i^2} > 0, \tag{47}$$

hence both factors in curly brackets in the denominator of (46) are positive. It remains to show that the curly bracket in the numerator is positive. To this end define

$$X_i^{\pm} = \frac{a_i^2}{1 \pm \lambda a_i^2} \tag{48}$$

and express the curly bracket in the numerator in terms of the bi-linear form,

$$X^+ . X^- := \frac{1}{(D-2)} \sum_{i,j=1}^{p} X_i^+ K_{ij} X_j^- . \tag{49}$$

K_{ij} here are the components of the $p \times p$ matrix

$$\mathbf{K} = (D-2)\mathbb{1} - \mathbf{I} \tag{50}$$

where $\mathbf{I}$ is the $p \times p$ all of whose entries are 1. The eigenvectors of $\mathbf{K}$ are the same as the eigenvectors of $\mathbf{I}$ and the latter has one eigenvalue equal to p and $p-1$ degenerate zero eigenvalues, hence $\mathbf{K}$ has one eigenvalue equal to $D-2-p = \frac{D-3+\epsilon}{2}$ and $p-1$ eigenvalues equal to $D-3$. All that concerns us here is that $\mathbf{K}$ is positive definite. We can use the identity

$$X^+ . X^- = \frac{1}{2}((X^+ + X^-).(X^+ + X^-) - X^+ . X^+ - X^- . X^-), \tag{51}$$

with $X_i^+ + X_i^- = \frac{2a_i^2}{1-\lambda^2 a_i^4}$, to arrive at

$$\begin{aligned} X^+ . X^- &= \sum_{i,j=1}^{p} \frac{(1-\lambda^2 a_i^2 a_j^2) a_i^2 a_j^2}{(1-\lambda^2 a_i^4)(1-\lambda^2 a_j^4)} K_{ij} \\ &\geq (1-\lambda^2 a_{\mathrm{Max}}^2) \sum_{i,j=1}^{p} \frac{a_i^2}{(1-\lambda^2 a_i^4)} K_{ij} \frac{a_j^2}{(1-\lambda^2 a_j^4)}, \end{aligned}$$

where $a_{\mathrm{Max}}^2 = \max(a_1^2, \dots a_p^2)$. Since all a_i satisfy $a_i^2 \leq \frac{1}{\lambda}$ we have $1 - \lambda^2 a_{\mathrm{Max}}^2 \geq 0$ and hence $X^+ . X^- \geq 0$. The compressibility is thus bounded from below.

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